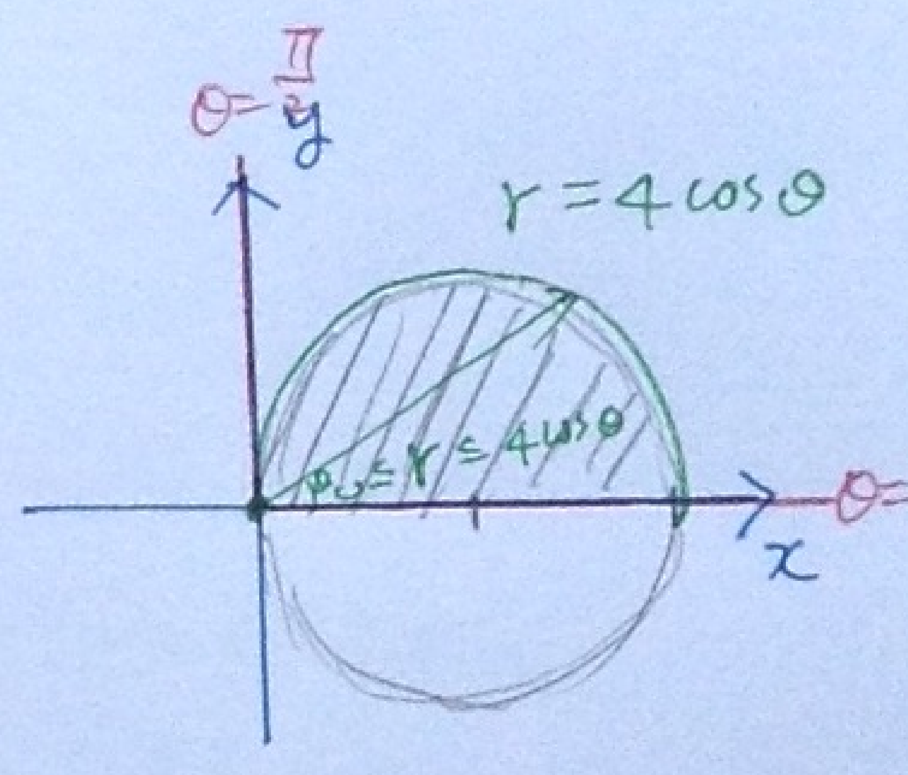
§4.3 $dA = r\,dr\,d\theta$

#6. Sketch the region whose area is given by the integral and evaluate the integral. $\int_0^{\frac{\pi}{2}} \int_0^{4\cos\theta} r\,dr\,d\theta$



sol: 1° $\begin{cases} 0 \le r \le 4\cos\theta \\ 0 \le \theta \le \frac{\pi}{2} \end{cases}$

先畫草圖 $r = 4\cos\theta$

$r^2 = 4r\cos\theta$

$x^2 + y^2 - 4x = 0$

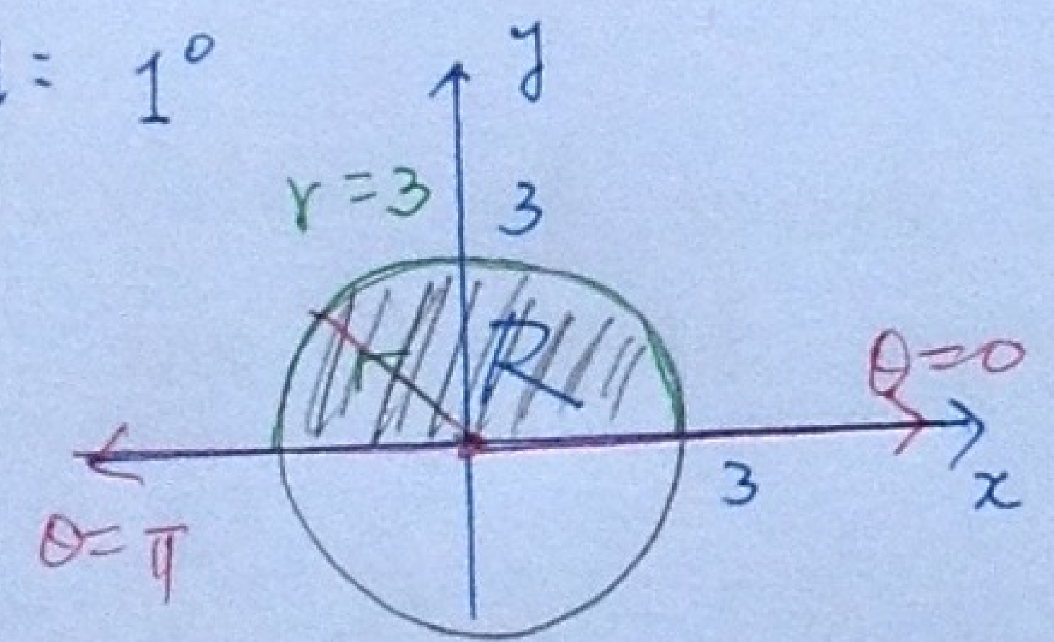
$(x-2)^2 + y^2 = 2^2$

2° 斜線部份（上半disk）即為所求，

故 area $= \int_0^{\frac{\pi}{2}} \frac{1}{2} r^2 \Big|_0^{4\cos\theta} d\theta = \int_0^{\frac{\pi}{2}} 8\cos^2\theta\, d\theta = 8 \times \frac{\pi}{4} = 2\pi$ ※

#9. $\iint_R \cos(x^2+y^2)\,dA$, where R: lies above the x-axis within the circle $x^2+y^2 = 9$

sol: 1°



用 polar coordinate describe R

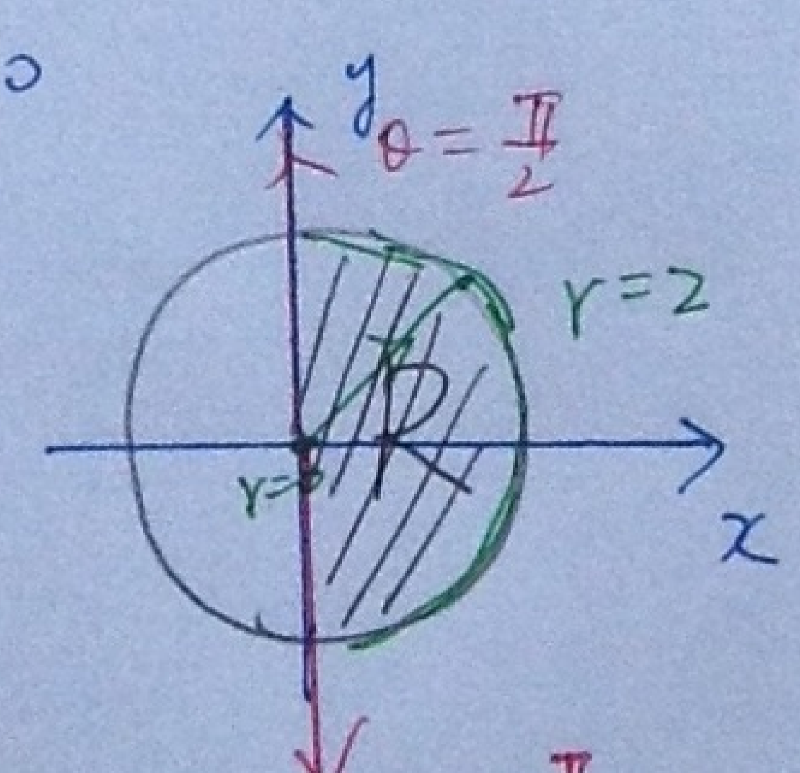
$\begin{cases} 0 \le \theta \le \pi \\ 0 \le r \le 3 \end{cases}$

2° $\iint_R \cos(x^2+y^2)\,dA = \int_0^{\pi} \int_0^3 \cos r^2 \underbrace{r\,dr}_{dr^2 \frac{1}{2}}\,d\theta$

$= \pi \left(\frac{1}{2} \sin r^2 \Big|_{r=0}^{r=3} \right) = \frac{\pi}{2} \sin 9$ ※

#10. $\iint_R \sqrt{4-x^2-y^2}\,dA$, $R = \{(x,y) \mid x^2+y^2 \le 4,\ x \ge 0\}$

sol: 1°



用 polar coordinate describe R

$\begin{cases} -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \\ 0 \le r \le 2 \end{cases}$

2° $\iint_R \sqrt{4-x^2-y^2}\,dA$

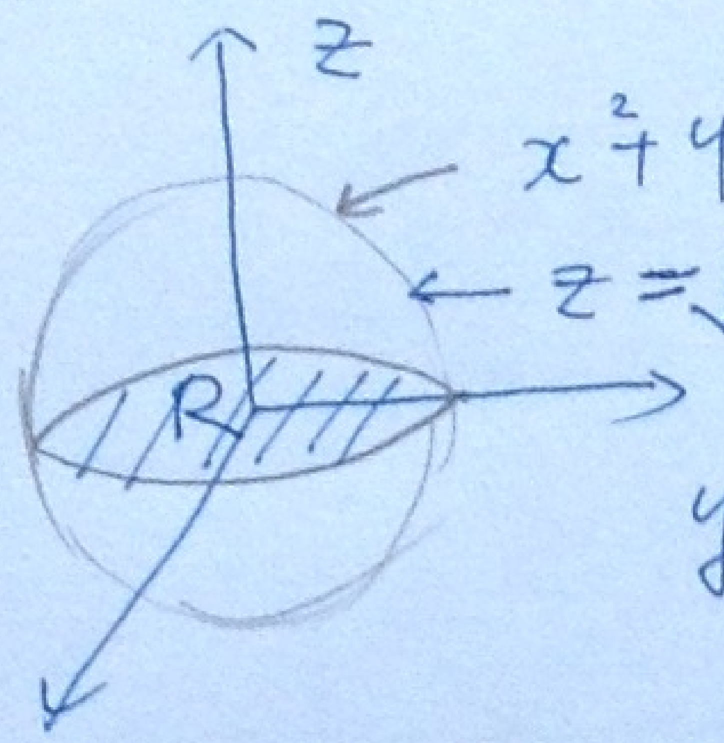
$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^2 \sqrt{4-r^2}\, \underbrace{r\,dr}_{\frac{1}{2} dr^2 = -\frac{1}{2} d(4-r^2)}\, d\theta$

$= \pi \left(-\frac{1}{2}\right) \cdot \left(\frac{2}{3}\right) \left[(4-r^2)^{\frac{3}{2}} \Big|_{r=0}^{r=2} \right] = \frac{8}{3}\pi$ ※

#15. a sphere of radius a 之 volume

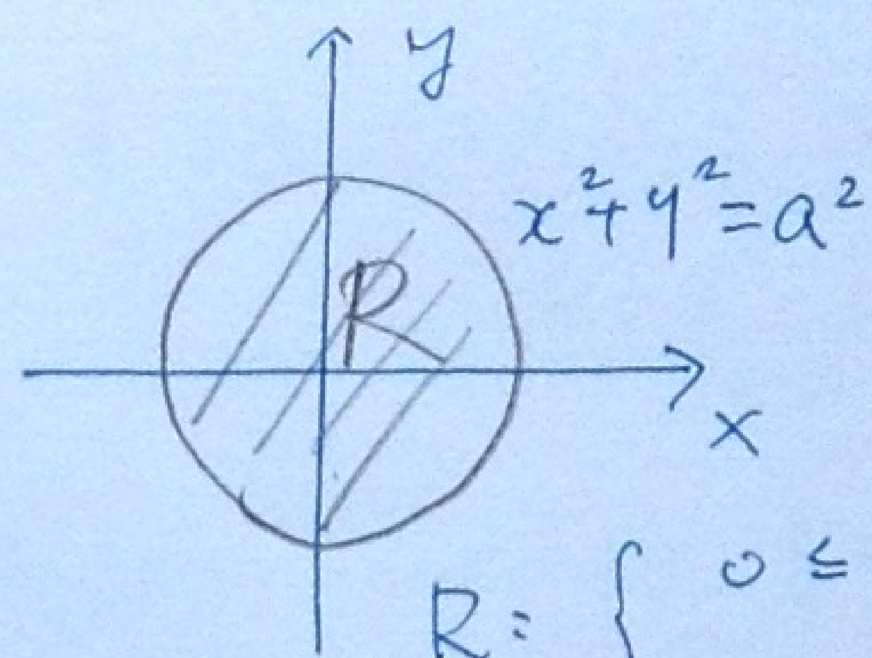
屋顶？地基？

sol: 1°



$x^2+y^2+z^2=a^2$

$z=\sqrt{a^2-x^2-y^2}$



$R: \begin{cases} 0 \le \theta \le 2\pi \\ 0 \le r \le a \end{cases}$

上半+下半球 → 2; 屋顶 ↑; R → 地基

$$2^\circ \quad V = 2\iint_R \sqrt{a^2-x^2-y^2}\,\underline{\underline{dA}} = 2\int_0^{2\pi}\int_0^a (a^2-r^2)^{\frac{1}{2}}\, \underline{r\,dr\,d\theta}$$

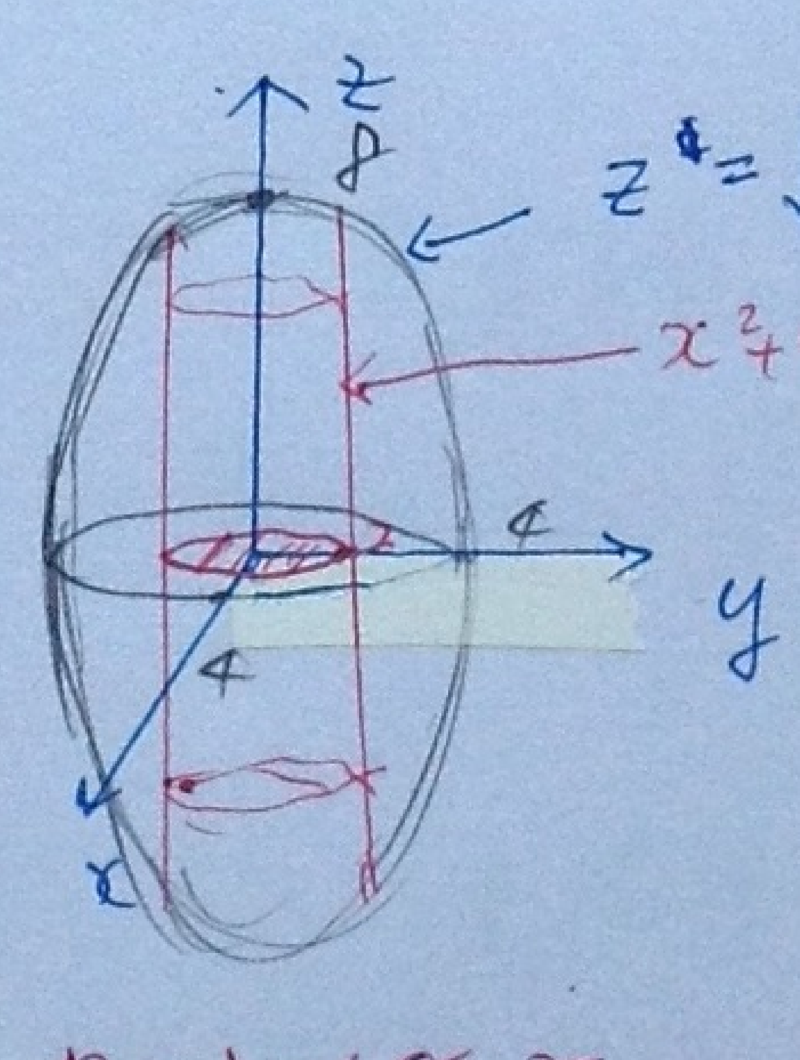
$$r\,dr = \tfrac{1}{2}dr^2 = -\tfrac{1}{2}d(a^2-r^2)$$

$$= -2\pi\,\frac{2}{3}(a^2-r^2)^{\frac{3}{2}}\Big|_{r=0}^{r=a} = \underline{\frac{4}{3}\pi a^3} \quad ※$$

(球体体積 $\frac{4}{3}\pi$ 半径^3)

#19. Volume of inside both the cylinder $x^2+y^2=4$ and the ellipsoid $4x^2+4y^2+z^2=64$

sol: 1° $\dfrac{x^2}{4^2}+\dfrac{y^2}{4^2}+\dfrac{y^2}{8^2}=1$



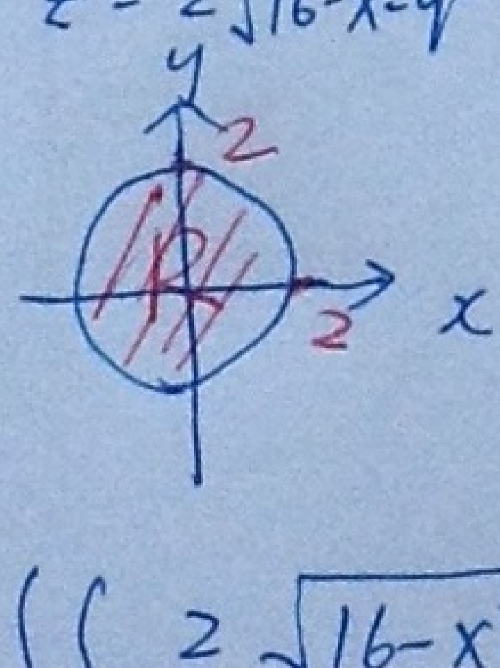
$z=\sqrt{64-4x^2-4y^2} = 2\sqrt{16-x^2-y^2}$

$x^2+y^2=4$

2° $V = 2V_上$

$V_上$ 屋顶 $z = 2\sqrt{16-x^2-y^2}$

地基



$R: \begin{cases} 0 \le \theta \le 2\pi \\ 0 \le r \le 2 \end{cases}$

$$3^\circ \quad V = 2\iint_R 2\sqrt{16-x^2-y^2}\,\underline{\underline{dA}} = 4\int_0^{2\pi}\int_0^2 \sqrt{16-r^2}\; r\,dr\,d\theta$$

$$r\,dr = -\tfrac{1}{2}d(16-r^2)$$

$$= -2\cdot\frac{2}{3}(16-r^2)^{\frac{3}{2}}\Big|_{r=0}^{2}\cdot 2\pi$$

$$= \underline{\frac{8}{3}\pi\,(64-24\sqrt{3})} \quad ※$$

§4.3 $dA = dx\,dy = dy\,dx = r\,dr\,d\theta$

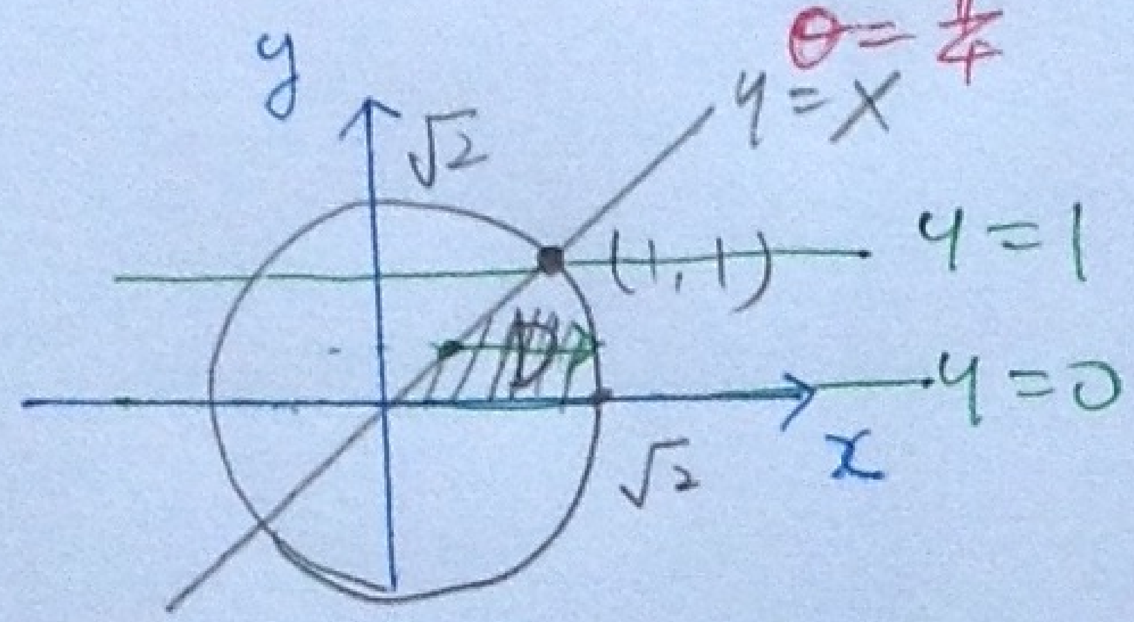
#25. $\displaystyle\int_0^1 \int_y^{\sqrt{2-y^2}} (x+y)\,dx\,dy =$

sol: 1°

$D: \begin{cases} y \le x \le \sqrt{2-y^2} \\ 0 \le y \le 1 \end{cases}$ 先畫出 $y=x$，及 $x^2 = 2-y^2$

交点 $x^2=1$

$x=\pm 1$



用 polar coordinate 重描寫 D

$D: \begin{cases} 0 \le \theta \le \frac{\pi}{4} \\ 0 \le r \le \sqrt{2} \end{cases}$

2° 原式 $\displaystyle = \int_0^{\frac{\pi}{4}} \int_0^{\sqrt{2}} r[\cos\theta + \sin\theta]\, \underbrace{r\,dr\,d\theta}_{dx\,dy \text{ or } dA}$

$\displaystyle = \int_0^{\frac{\pi}{4}} [\cos\theta + \sin\theta]\,d\theta \cdot \int_0^{\sqrt{2}} r^2\,dr = \frac{2\sqrt{2}}{3}$ #

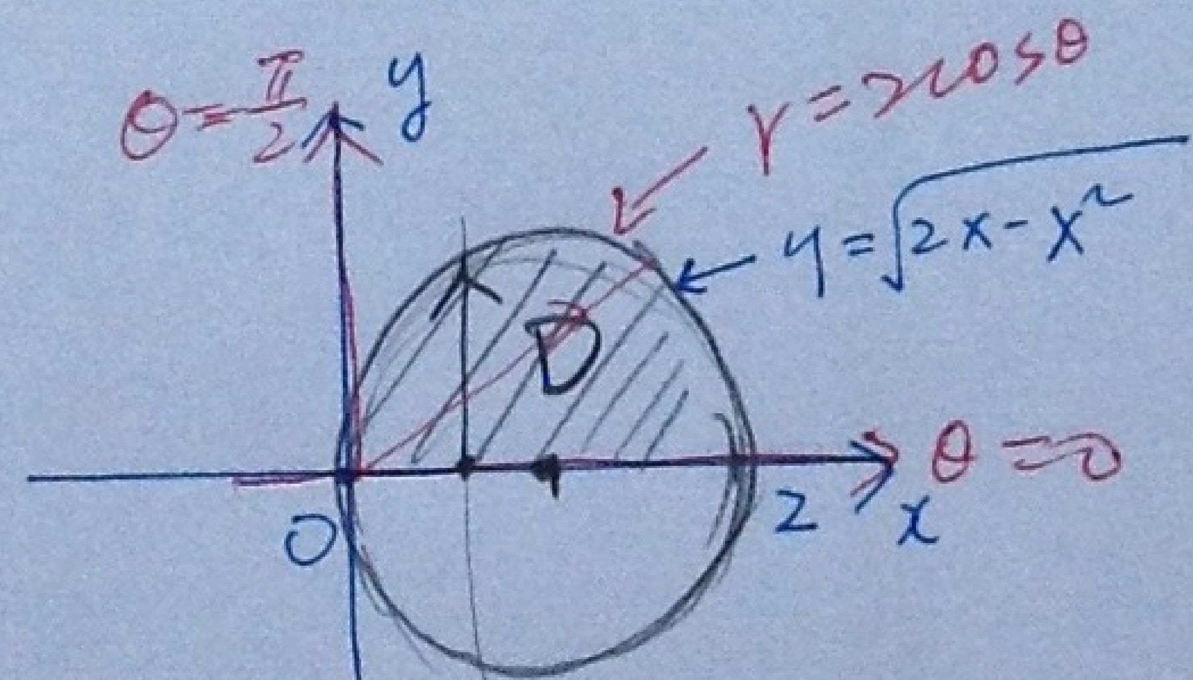
#26. $\displaystyle\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\,dy\,dx$

sol: 1°

$D: \begin{cases} 0 \le y \le \sqrt{2x-x^2} \\ 0 \le x \le 2 \end{cases}$ 先畫出 $y^2 = 2x - x^2$

$\boxed{x^2+y^2-2x=0}$

$(x-1)^2+y^2=1$



重描寫 D 用 polar coor.

$r^2 - 2r\cos\theta = 0$

$\Rightarrow \underline{r = 2\cos\theta}$ or $r=0$ (已含 $r=2\cos\theta$ 中)

$D: \begin{cases} 0 \le \theta \le \frac{\pi}{2} \\ 0 \le r \le 2\cos\theta \end{cases}$

2° 原式 $\displaystyle = \int_0^{\frac{\pi}{2}} \int_0^{2\cos\theta} r\, \underbrace{r\,dr\,d\theta}_{dy\,dx} = \int_0^{\frac{\pi}{2}} \left(\frac{1}{3} r^3 \Big|_0^{2\cos\theta}\right) d\theta$

$\displaystyle = \frac{8}{3} \int_0^{\frac{\pi}{2}} \underbrace{\cos^3\theta\,d\theta}_{\cos^2\theta\,(\cos\theta)\,d\theta} = \frac{8}{3} \int_{\theta=0}^{\theta=\frac{\pi}{2}} (1-\sin^2\theta)\,d\sin\theta = \frac{8}{3}\left(\sin\theta - \frac{1}{3}\sin^3\theta\right)\Big|_{\theta=0}^{\theta=\frac{\pi}{2}}$

$\cos^2\theta = 1-\sin^2\theta$, $\cos\theta\,d\theta = d\sin\theta$

$\displaystyle = \frac{8}{3} \times \frac{2}{3} = \frac{16}{9}$ #